# Problem Set 6

due Sunday 28th of Azar in class

**1-** This problem offers a different derivation of the Green's functions of the wave equation ($c = 1$ in this problem).
**(a)** To begin, consider the homogeneous equation

$$(\nabla^2 - \partial_t^2)\Psi(\mathbf{x}, t) = 0. \tag{1}$$

Show that the Fourier transform $\tilde{\Psi}(\mathbf{k}, \omega)$ of $\Psi$ satisfies $(-k^2 + \omega^2)\tilde{\Psi} = 0$ and its solution is

$$\tilde{\Psi}(\mathbf{k}, \omega) = f(\omega)\delta(\omega - k) + g(\omega)\delta(\omega + k), \tag{2}$$

where $f$ and $g$ are non-singular functions of $\omega \mp k$, i.e., $\lim_{\omega \to k}(\omega - k)f(\omega) = 0$ and $\lim_{\omega \to -k}(\omega + k)g(\omega) = 0$.
**(b)** The Green's functions satisfy $(\nabla^2 - \partial_t^2)G(\mathbf{x}, t) = \alpha\delta(\mathbf{x})\delta(t)$. With Jackson's convention $\alpha = -4\pi$; but other choices like $\alpha = \pm 1$ or $\pm i$ are also common. Show that the general Green's function is

$$\tilde{G}(\mathbf{k}, \omega) = \frac{\alpha}{\omega^2 - k^2} + f(\omega)\delta(\omega - k) + g(\omega)\delta(\omega + k), \tag{3}$$

where the convention for Fourier transformation is

$$\tilde{G}(\mathbf{k}, \omega) = \int dt d^3x G(\mathbf{x}, t) e^{i\omega t - i\mathbf{k}\cdot\mathbf{x}}, \qquad G(\mathbf{x}, t) = \int \frac{d\omega d^3k}{(2\pi)^4} \tilde{G}(\mathbf{k}, \omega) e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}. \tag{4}$$

**(c)** The integral $\int d\omega e^{-i\omega t}/(\omega^2 - k^2)$ depends on how to approach $\omega = k$. We do it symmetrically, i.e., use the so-called Cauchy principal value,

$$P\int_{-\infty}^{\infty} \frac{e^{-i\omega t} d\omega}{\omega^2 - k^2} := \lim_{\kappa \to 0+} \left[\int_{-\infty}^{-k-\kappa} + \int_{-k+\kappa}^{k-\kappa} + \int_{k+\kappa}^{\infty}\right] \frac{e^{-i\omega t} d\omega}{\omega^2 - k^2}. \tag{5}$$

Now remember the formula (prove it if you have not seen it before)

$$\frac{1}{\omega - k \pm i\eta} = P\frac{1}{\omega - k} \mp i\pi\delta(\omega - k), \tag{6}$$

which is valid for infinitesimal $\eta > 0$, when integrated over the real $\omega$ axis. Show that we can write

$$\begin{aligned} P\frac{1}{\omega^2 - k^2} &= \frac{1}{\omega^2 - k^2 \pm i\epsilon} \pm \frac{i\pi}{2k}\left[\delta(\omega - k) + \delta(\omega + k)\right] &(7)\\ &= \frac{1}{\omega^2 - k^2 \pm i\epsilon\omega} \pm \frac{i\pi}{2k}\operatorname{sgn}(\omega)\left[\delta(\omega - k) + \delta(\omega + k)\right], &(8) \end{aligned}$$

where $\epsilon > 0$ is infinitesimal.
**(d)** Find four suitable pairs of $(f, g)$ in Eq. (3) to obtain the following four Green's functions:[1]

$$G_\pm(\mathbf{x}, t) = \alpha\int \frac{d\omega d^3k}{(2\pi)^4} \frac{e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{\omega^2 - k^2 \pm i\epsilon}, \qquad G^\pm(\mathbf{x}, t) = \alpha\int \frac{d\omega d^3k}{(2\pi)^4} \frac{e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{\omega^2 - k^2 \pm i\epsilon\omega}. \tag{9}$$

***(e)** Show that

$$G_\pm(-\mathbf{x}, t) = G_\pm(\mathbf{x}, t), \qquad G_\pm(\mathbf{x}, -t) = G_\pm(\mathbf{x}, t), \qquad G^\pm(-\mathbf{x}, t) = G^\pm(\mathbf{x}, t), \qquad G^\pm(\mathbf{x}, -t) = G^\mp(\mathbf{x}, t), \tag{10}$$

and

$$\Delta(-\mathbf{x}, t) = \Delta(\mathbf{x}, t), \qquad \Delta(\mathbf{x}, -t) = -\Delta(\mathbf{x}, t), \qquad \partial_t\Delta(\mathbf{x}, t)|_{t=0} = -\delta(\mathbf{x}), \tag{11}$$

where $\Delta = (G^+ - G^-)/\alpha$ is called Schwinger's $\Delta$-function.
**(f)** The left column of Figure 1 shows the location of the poles in the four integrals of Eq. (9) in the complex $\omega$ plane. All integrations are over the real line. Convince yourself that these integrals can be converted to

$$\int \frac{d^3k}{(2\pi)^3} \int_C \frac{d\omega}{2\pi} \frac{e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}}{\omega^2 - k^2}, \tag{12}$$

[1] $G_+$ is usually denoted by $G_F$ and is called Feynman (or causal, or time-ordered) Green's function. $G_- = G_D$ is called anti-time-ordered, acausal, or Dyson Green's function. $G^+$ is usually denoted by $G_R$ and known as the retarded Green's function. $G^- = G_A$ is the advanced Green's function.

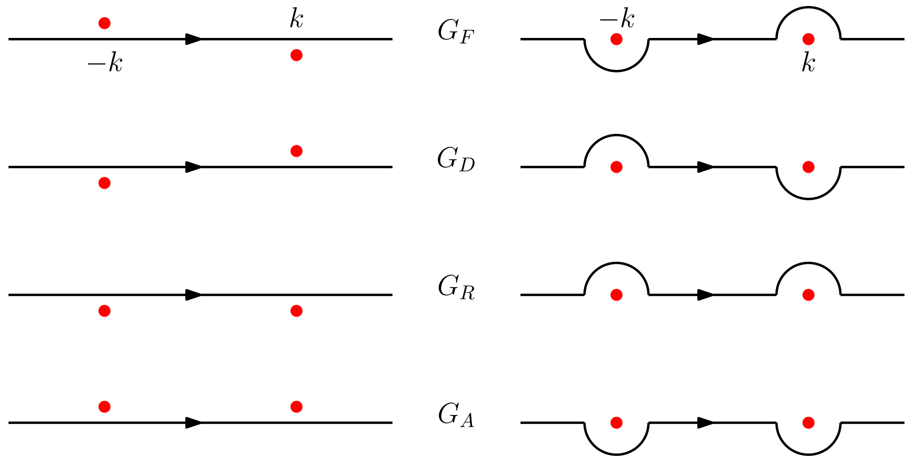

Figure 1: Integration contours and poles (red dots) in the complex $\omega$ plane.

where $C$ is the corresponding $\omega$ contour given in the right column of Figure 1. Clearly, $G_+ - G_-$ and $G^+ - G^-$ satisfy the homogeneous wave equation (why?). What are the corresponding contours?
**(g)** Now use the contour on the right column of Figure (1) and Cauchy's residue theorem to show that

$$G^{\pm} = \frac{-\alpha}{4\pi r}\delta(t \mp r) = \frac{-\alpha}{2\pi}\theta(\pm t)\delta(t^2 - r^2), \tag{13}$$

where $\theta$ is the unit step function.

**2-** [Jackson 6.2] The charge and current densities for a single point charge $q$ can be written formally as

$$\rho(\mathbf{x}', t') = q\delta[\mathbf{x}' - \mathbf{r}(t')]; \qquad \mathbf{J}(\mathbf{x}', t') = q\mathbf{v}(t')\delta[\mathbf{x}' - \mathbf{r}(t')] \tag{14}$$

where $\mathbf{r}(t')$ is the charge's position at time $t'$ and $\mathbf{v}(t')$ is its velocity. In evaluating expressions involving the retarded time, one must put $t' = t_{\text{ret}} = t - R(t')/c$, where $\mathbf{r} = \mathbf{x} - \mathbf{r}(t')$ (but $\mathbf{r} = \mathbf{x} - \mathbf{x}'(t')$ inside the delta functions).
**(a)** As a preliminary to deriving the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge, show that

$$\int d^3x' \delta[\mathbf{x}' - \mathbf{r}(t_{\text{ret}})] = \frac{1}{\kappa} \tag{15}$$

where $\kappa = 1 - \mathbf{v} \cdot \hat{\mathbf{r}}/c$. Note that $\kappa$ is evaluated at the retarded time.
**(b)** Starting with the Jefimenko generalizations of the Coulomb and Biot-Savart laws, use the expressions for the charge and current densities for a point charge and the result of part (a) to obtain the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{r}}}{\kappa R^2}\right]_{\text{ret}} + \frac{\partial}{c\partial t}\left[\frac{\hat{\mathbf{r}}}{\kappa R}\right]_{\text{ret}} - \frac{\partial}{c^2\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{\text{ret}}\right\} \tag{16}$$

and

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{r}}}{\kappa R^2}\right]_{\text{ret}} + \frac{\partial}{c\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{r}}}{\kappa R}\right]_{\text{ret}}\right\} \tag{17}$$

**(c)** In our notation Feynman's expression for the electric field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{r}}}{R^2}\right]_{\text{ret}} + \frac{[R]_{\text{ret}}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{r}}}{R^2}\right]_{\text{ret}} + \frac{\partial^2}{c^2\partial t^2}\left[\hat{\mathbf{r}}\right]_{\text{ret}}\right\} \tag{18}$$

while Heaviside's expression for the magnetic field is

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{r}}}{\kappa^2 R^2}\right]_{\text{ret}} + \frac{1}{c[R]_{\text{ret}}}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{r}}}{\kappa}\right]_{\text{ret}}\right\} \tag{19}$$

Show the equivalence of the two sets of expressions for the fields.

References: O. Heaviside, *Electromagnetic Theory*, Vol. 3 (1912), p. 464, Eq. (214). R. P. Feynman, *The Feynman Lectures in Physics*, Vol. 1 (1963), Chapter 28, Eq. (28.3).