599
Electrodynamics 1

Problem Set 5

due Sunday 14th of Azar in class

1- Let $\rho(\mathbf{x},t)$ denote the instantaneous charge density and $\mathbf{v}(\mathbf{x},t) = \mathbf{J}(\mathbf{x},t)/\rho(\mathbf{x},t)$ be the velocity field of the charge distribution.
(a) In electrostatics, Maxwell's equations read

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \qquad \nabla \cdot \mathbf{E} = 0, \qquad \mathbf{B} = 0. \tag{1}$$

Using the full set of Maxwell's equations show that a sufficient condition for this is $\partial\rho/\partial t = \mathbf{v} = 0$, or equivalently, $\mathbf{J} = 0$.
(b) In magnetostatics, Maxwell's equations read

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \qquad \nabla \times \mathbf{E} = 0, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{2}$$

Show that a sufficient condition for this is $\partial\rho/\partial t = \partial\mathbf{J}/\partial t = 0$, or equivalently, $\nabla \cdot \mathbf{J} = \partial\mathbf{v}/\partial t = 0$.

2- [Jackson 5.1] Starting with the differential expression

$$d\mathbf{B} = \frac{\mu_0 I}{4\pi} d\mathbf{l}' \times \frac{\mathbf{x}-\mathbf{x}'}{|\mathbf{x}-\mathbf{x}'|^3} \tag{3}$$

for the magnetic induction at the point P with coordinate $\mathbf{x}$ produced by an increment of current $Id\mathbf{l}'$ at $\mathbf{x}'$, show explicitly that for a closed loop carrying a current I the magnetic induction at P is

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \nabla\Omega \tag{4}$$

where Ω is the solid angle subtended by the loop at the point P. This corresponds to a magnetic scalar potential, $\Phi_M = -\mu_0 I \Omega/4\pi$. The sign convention for the solid angle is that Ω is positive if the point P views the "inner" side of the surface spanning the loop, that is, if a unit normal $\mathbf{n}$ to the surface is defined by the direction of current flow via the right-hand rule, Ω is positive if $\mathbf{n}$ points *away* from the point P, and negative otherwise. This is the same convention as in Section 1.6 for the electric dipole layer.

3- [Zangwill 11.2] **Origin Independence of Magnetic Multipole Moments** Let $\mathbf{j}(\mathbf{r})$ be an arbitrary current distribution.
(a) Show that the components of the magnetic dipole moment $\mathbf{m} = \frac{1}{2}\int d^3r \mathbf{r} \times \mathbf{j}$ are invariant to a rigid shift of the origin of coordinates.

4- [Zangwill 11.14] **No Magnetic Dipole Moment** Show that a current density with vector potential $\mathbf{A}(\mathbf{r}) = f(\mathbf{r})\mathbf{r}$ has zero magnetic dipole moment.

5- [Jackson 5.15] Consider two long, straight wires, parallel to the z axis, spaced a distance d apart and carrying currents I in opposite directions. Describe the magnetic field $\mathbf{H}$ in terms of a magnetic scalar potential Φ_M, with $\mathbf{H} = -\nabla\Phi_M$.
(a) If the wires are parallel to the z axis with positions, $x = \pm d/2$, $y = 0$, show that in the limit of small spacing, the potential is approximately that of a two-dimensional dipole,

$$\Phi_M \approx -\frac{Id\sin\phi}{2\pi\rho} + O(d^2/\rho^2) \tag{5}$$

where ρ and ϕ are the usual polar coordinates.
(b) The closely spaced wires are now centered in a hollow right circular cylinder of steel, of inner (outer) radius a (b) and magnetic permeability $\mu = \mu_r\mu_0$. Determine the magnetic scalar potential in the three regions, $0 < \rho < a$, $a < \rho < b$, and $\rho > b$. Show that the field outside the steel cylinder is a two-dimensional dipole field, as in part (a), but with a strength reduced by the factor

$$F = \frac{4\mu_r b^2}{(\mu_r+1)^2 b^2 - (\mu_r-1)^2 a^2}. \tag{6}$$

[...]
(c) Assuming that $\mu_r \gg 1$, and $b = a + t$, where the thickness $t \ll b$, write down an approximate expression for F and determine its numerical value for $\mu_r = 200$ (typical of steel at 20 G), $b = 1.25$ cm, $t = 3$ mm. The shielding effect is relevant

for reduction of stray fields in residential and commercial 60 Hz, 110 or 220 V wiring. The figure [in Jackson] illustrates the shielding effect for $a/b = 0.9$, $\mu_r = 100$.

6- [Zangwill 13.12] **A Dipole in a Magnetizable Sphere** A point magnetic dipole is located at the center of a magnetizable sphere with radius R and permeability μ. Find $\mathbf{H}(\mathbf{r})$ everywhere.

7- In this problem you're going to obtain the hysteresis loop in the $\mu_0 H_{\text{in}}$-B_{in} plane from the one in the B_0-M plane. [I am using the notation of section 5.11 in Jackson.] We are more familiar with the B_0-M plane: as we increase the external field B_0 the magnetization M increases until it reaches a saturation point after which further increase in B_0 doesn't change M. So we have something like figure 5.12 in Jackson (except that the labels are renamed: $\mu_0 H \to B_0$ and $B \to M$) where far to the right M asymptotes to a constant value on a horizontal line (not shown thoroughly in figure 5.12). Now use this diagram and Eqs. (5.112) to obtain the loop in the $\mu_0 H_{\text{in}}$-B_{in} plane. Also find the initial trajectory which joins the origin ($B_{\text{in}} = 0, H_{\text{in}} = 0$) to the saturation point. Explain the difference in the slope of the asymptote between the plots in the two planes.

You can use computer software for the plots. You may model the loop curves by $M = c\tanh(a(B_0 \pm b))$ (one for the upper and one for the lower curve) and the initial trajectory by $M = c\tanh(aB_0)$. To find realistic values for a, b, and c, you should note that the two axes of figure 5.12 (the $\mu_0 H_{\text{in}}$-B_{in} plane) have scales that differ by $\sim 10^4$.