Gravitation 1 PS10

1) d'Inverno (12.67) $\frac{1}{2}\epsilon\,\Box\left(h_{\mu\nu}-\frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}\right) = -\kappa_n T_{\mu\nu}$ (*)

Contracting with $\eta^{\mu\nu}$: $\frac{1}{2}\epsilon\,\Box\left(\eta^{\mu\nu}h_{\mu\nu}-\frac{1}{2}\underbrace{\eta^{\mu\nu}\eta_{\mu\nu}}_{=(n+1)}\eta^{\rho\sigma}h_{\rho\sigma}\right) = -\kappa_n\eta^{\mu\nu}T_{\mu\nu}$

$$\Rightarrow \frac{1}{2}\epsilon\,\Box\left(\frac{1}{2}(1-n)\,\eta^{\mu\nu}h_{\mu\nu}\right) = -\kappa_n\eta^{\mu\nu}T_{\mu\nu} \quad \Rightarrow -\frac{1}{4}\epsilon\,\Box(\eta^{\mu\nu}h_{\mu\nu}) = \frac{\kappa_n}{1-n}\eta^{\mu\nu}T_{\mu\nu}$$

$$\xrightarrow{(*)} \quad \frac{1}{2}\epsilon\,\Box h_{\mu\nu} + \eta_{\mu\nu}\underbrace{\left(-\frac{1}{4}\epsilon\,\Box(\eta^{\rho\sigma}h_{\rho\sigma})\right)}_{=\frac{\kappa_n}{1-n}\eta^{\rho\sigma}T_{\rho\sigma}} = -\kappa_n T_{\mu\nu}$$

$$\Rightarrow \quad \frac{1}{2}\epsilon\,\Box h_{\mu\nu} = -\kappa_n\left(T_{\mu\nu}-\frac{1}{n-1}\eta_{\mu\nu}\eta^{\rho\sigma}T_{\rho\sigma}\right)$$

Slow-motion approximation: $\partial_t \sim \epsilon\,\partial_i \quad \Rightarrow \epsilon\,\Box h_{\mu\nu} = \epsilon(-\partial_t\partial_t+\nabla^2)h_{\mu\nu} \sim \epsilon\nabla^2 h_{\mu\nu}$

The energy-momentum tensor for a non-relativistic fluid: $T_{\mu\nu} = \rho\,\delta^0_\mu\delta^0_\nu \Rightarrow \begin{cases} T_{00} = \rho \\ \eta^{\mu\nu}T_{\mu\nu} = -\rho \end{cases}$

$\rightsquigarrow$ The zero-zero component: $\frac{1}{2}\epsilon\nabla^2 h_{00} = -\kappa_n\left(\rho-\frac{1}{n-1}\rho\right)$

$$\epsilon h_{00} = g_{00}-\eta_{00} \quad \Rightarrow \frac{1}{2}\nabla^2 g_{00} = -\kappa_n\frac{n-2}{n-1}\rho \quad \Rightarrow \boxed{\nabla^2 g_{00} = -2\frac{n-2}{n-1}\kappa_n\rho}\ \checkmark$$

To find the unit of κ_n, note that $S = -\frac{1}{2}\kappa_n^{-1}\int d^{n+1}x\,\sqrt{-g}\,R$, and $[S]=1$

Let's assume all the coordinates have the unit of length: $[x^\mu] = L$

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu \qquad [ds^2] = L^2 \Rightarrow [g_{\mu\nu}] = [g^{\mu\nu}] = 1 \quad \Rightarrow [\sqrt{-g}] = 1$$

$$R = g^{\mu\nu}R_{\mu\nu} = g^{\mu\nu}R^\alpha{}_{\mu\alpha\nu} \Rightarrow [R] = [\Gamma^\alpha_{\beta\gamma}]^2 = [g^{-}\partial_\mu g_{-}]^2 = [\partial_\mu]^2 = L^{-2}$$

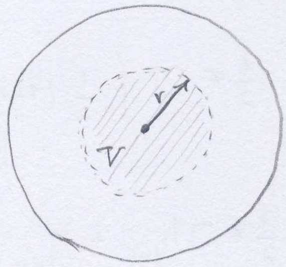
$$\Rightarrow [S] = 1 = [\kappa_n]^{-1}L^{n+1}.\,L^{-2} \quad \Rightarrow [\kappa_n] = L^{n-1} = [M_n]^{1-n}$$

2) a) We obtained for a n-dimensional space: $\nabla^2 g_{00} = -2\,\frac{n-2}{n-1} K_n \rho$

If we put $g_{00} = -1 - 2\,\frac{n-2}{n-1}\,\frac{K_n}{G_n^\phi}\,\phi$ in the above equation:

$$\nabla^2\left(-2\,\frac{n-2}{n-1}\,\frac{K_n}{G_n^\phi}\,\phi\right) = -2\,\frac{n-2}{n-1} K_n \rho \quad \Rightarrow \quad \nabla^2\phi = G_n^\phi \rho \quad \checkmark$$

b) A spherically symmetric mass, with mass density $\rho(r)$:



$$\nabla^2\phi = G_n^\phi \rho$$

Integrate over V $\Rightarrow$

$$\int_V d^n\vec{x}\ \underbrace{\nabla^2\phi}_{\partial^i\partial_i\phi} = G_n^\phi \int_V d^n\vec{x}\,\rho = G_n^\phi\, \underbrace{M(r)}_{\text{enclosed mass}}$$

Gauss' $\Rightarrow$

$$\int_A d^{n-1}\vec{x}\ \hat{n}\cdot\vec{\nabla}\phi = G_n^\phi M(r)$$

$\longrightarrow$ Due to the spherical symmetry, $\vec{\nabla}\phi$ is constant over the $(n-1)$-d surface.

$$\Rightarrow \int d^{n-1}\vec{x} = A_{n-1} r^{n-1}, \quad \underbrace{\hat{n}}_{\hat{r}}\cdot\vec{\nabla}\phi = \frac{d\phi}{dr} \quad \Rightarrow \quad A_{n-1} r^{n-1} \frac{d\phi}{dr} = G_n^\phi M(r)$$

$\hookrightarrow$ ϕ is a function of r only.

rename $r \to r'$

$$\Rightarrow \frac{d\phi(r')}{dr'} = \frac{G_n^\phi}{A_{n-1}}\,\frac{M(r')}{r'^{\,n-1}} \xrightarrow[\text{over } r']{\text{integrate}} \int_\infty^r dr'\,\frac{d\phi(r')}{dr'} = \frac{G_n^\phi}{A_{n-1}} \int_\infty^r dr'\,\frac{M(r')}{r'^{\,n-1}}$$

$\hookrightarrow$ reference point for potential

Conventionally, $\phi(r\to\infty) \to 0$

$$\Rightarrow \phi(r) = \frac{G_n^\phi}{A_{n-1}} \int_\infty^r dr'\,\frac{M(r')}{r'^{\,n-1}}$$

{ On the other hand, you may start from eqn. (5), and check $\nabla^2\phi = G_n^\phi \rho$:

First, note that the Laplace operator in spherical coordinate acting on a function of r in n-dimension is

$$\nabla^2 f(r) = \frac{1}{r^{n-1}}\frac{\partial}{\partial r}\left(r^{n-1}\frac{\partial f}{\partial r}\right) = \frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{df}{dr}\right)$$

$$\Rightarrow \nabla^2\phi(r) = \frac{1}{r^{n-1}}\frac{d}{dr}\left(r^{n-1}\frac{G_n^\phi}{A_{n-1}}\frac{d}{dr}\int_\infty^r dr'\,\frac{M(r')}{r'^{\,n-1}}\right) = \frac{1}{r^{n-1}}\frac{d}{dr}\left(\frac{G_n^\phi}{A_{n-1}}\, r^{n-1}\,\frac{M(r)}{r^{n-1}}\right)$$

$$= \frac{1}{r^{n-1}}\frac{d}{dr}\left(G_n^\phi \int_0^r dr'\,\rho(r')\,r'^{\,n-1}\right) = G_n^\phi \frac{1}{r^{n-1}}\left(\rho(r)\, r^{n-1}\right) = G_n^\phi\, \rho(r) \quad \checkmark$$

}

▶ A little more about spherical surfaces and volumes in an n-dimensional space :

The metric for an n-dimensional sphere (S^{n-1} embeded in n-dimension) is :

$$ds^2 = dr^2 + r^2 \left[d\theta_{n-1}^2 + \sin^2\theta_{n-1} \left(d\theta_{n-2}^2 + \sin^2\theta_{n-2} \left(\cdots \left(d\theta_1^2 \right) \cdots \right) \right) \right] \qquad \begin{cases} 0 \leq \theta_1 < 2\pi \\ 0 \leq \theta_k \leq \pi & k > 1 \end{cases}$$

$$\Rightarrow \sqrt{|g|} = r^{n-1} \left(\sin\theta_{n-1} \right)^{n-2} \left(\sin\theta_{n-2} \right)^{n-3} \cdots \sin\theta_2$$

$$\Rightarrow \begin{cases} \text{Volume of a ball of radius } r : \quad V_n = \int_0^r dr' \, r'^{n-1} \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \cdots \int_0^{\pi} d\theta_{n-1} \left(\sin\theta_{n-1} \right)^{n-2} \\ \text{Surface of a ball of radius } r : \quad r^{n-1} \times \int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \cdots \int_0^{\pi} d\theta_{n-1} \left(\sin\theta_{n-1} \right)^{n-2} \end{cases}$$

$$= r^{n-1} \times 2\pi \times \prod_{k=2}^{n-1} \int_0^{\pi} d\theta_k \left(\sin\theta_k \right)^{k-1} \equiv r^{n-1} A_{n-1}$$

One could calculate A_{n-1} by induction, or with Mathematica!

$$\Rightarrow A_{n-1} = 2\pi \times \frac{\pi^{-1+\frac{n}{2}}}{\Gamma(\frac{n}{2})} = \frac{2\pi^{n/2}}{\Gamma(\frac{n}{2})} \checkmark \qquad \Rightarrow \int_{\substack{(n-1)-d \\ \text{hypersurface}}} dA = r^{n-1} A_{n-1}$$

Now, for the mass below raduis r :

$$M(r) = \int_{\text{hyper volume}} \underbrace{d^n\vec{x}}_{dV} \, \rho(\vec{x}) = \int_0^r dr' \, r'^{n-1} \rho(r') \underbrace{\int_0^{2\pi} d\theta_1 \int_0^{\pi} d\theta_2 \sin\theta_2 \cdots \int_0^{\pi} d\theta_{n-1} \sin\theta_{n-1}^{n-2}}_{\equiv A_{n-1}}$$

$$\Rightarrow M(r) = A_{n-1} \int_0^r dr' \, r'^{n-1} \rho(r')$$

In the case of $n = 3$:

$$A_2 = \frac{2\pi^{3/2}}{\Gamma(\frac{3}{2})} = \frac{2\pi^{3/2}}{\frac{1}{2}\sqrt{\pi}} = 4\pi$$

$$\Gamma(1+z) = z\,\Gamma(z)$$
$$\Gamma(1/2) = \sqrt{\pi}$$

$$\Rightarrow M(r) = 4\pi \int_0^r dr' \, r'^2 \rho(r') \quad \rightarrow \text{the familiar relation}$$

c) Assume the raduis of mass is R $\Rightarrow M_{total} = A_{n-1} \int_0^R dr \, r^{n-1} \rho(r)$

$\Rightarrow$ for $r > R$, M is a constant :

$$\phi(r)\Big|_{r>R} = \frac{G_n^{\phi}}{A_{n-1}} M_{tot} \int_{\infty}^{r} dr' \, r'^{1-n} = \frac{G_n^{\phi} M_{tot}}{A_{n-1}} \frac{1}{2-n} \, r'^{2-n} \Big|_{\infty}^{r}$$

$$n > 2 \Rightarrow r'^{2-n}\Big|_{\infty} = 0 \quad \Rightarrow \quad \phi(r>R) = -\frac{1}{n-2} \frac{G_n^{\phi} M_{tot}}{A_{n-1}} \frac{1}{r^{n-2}} \equiv -\frac{G_n^R M_{tot}}{r^{n-2}}$$

$$\Rightarrow G_n^{\phi} = (n-2) A_{n-1} G_n^R \checkmark$$

3) a) In the comoving frame (which the fluid is static in), $\frac{dx^i}{d\tau} = 0$ for $i = 1, 2, 3, \dots, n$

so the velocity (n+1)-vector has only the zero component : $u^\mu = (u^0, 0, 0, 0, \dots)$

We also know: $u^\mu u_\mu = -1 \Rightarrow g_{\mu\nu} u^\mu u^\nu = -1 \Rightarrow g_{00}(u^0)^2 = -1$

$$\Rightarrow -e^{\nu(r)} (u^0)^2 = -1 \Rightarrow u^0 = e^{-\nu/2} \Rightarrow u^\mu = (e^{-\nu/2}, 0, 0, \dots)$$ ✓

$$T^\mu{}_\nu = g_{\alpha\nu} T^{\mu\alpha} = (\rho(r) + p(r))\, g_{\alpha\nu} u^\mu u^\alpha + p(r)\, \delta^\mu_\nu$$

$$\Rightarrow T^0{}_0 = (\rho(r) + p(r))\, g_{\alpha 0} u^0 u^\alpha + p(r)$$ Since the metric is diagonal, α could only be 0.

$$\Rightarrow T^0{}_0 = (\rho(r) + p(r))(-e^\nu)(e^{-\nu/2})^2 + p(r) \Rightarrow \boxed{T^0{}_0 = -\rho(r)}$$ $u^1 = 0 \Rightarrow T^1{}_1 = p(r)$

b) $G^0{}_0 = \kappa_n T^0{}_0 \Rightarrow \frac{n-1}{2} \frac{1}{r^{n-1}} \frac{d}{dr}\left[r^{n-2}(e^{-\lambda} - 1)\right] = -\kappa_n \rho(r) \xrightarrow{r \to r'} \frac{d}{dr'}\left[r'^{n-2}(e^{-\lambda(r')} - 1)\right] = \frac{-2\kappa_n}{n-1} r'^{n-1} \rho(r')$

$\xrightarrow{\text{integrate}} \int_0^r dr' \frac{d}{dr'}\left[r'^{n-2}(e^{-\lambda(r')} - 1)\right] = \frac{-2\kappa_n}{n-1} \int_0^r dr'\, r'^{n-1} \rho(r') \Rightarrow r^{n-2}(e^{-\lambda} - 1) = \frac{-2\kappa_n}{n-1} \frac{M(r)}{A_{n-1}}$

$$\Rightarrow e^{-\lambda} - 1 = \frac{-2\kappa_n}{n-1} \frac{M(r)}{A_{n-1}} \times \frac{1}{r^{n-2}} \Rightarrow e^{\lambda(r)} = \left[1 - \frac{2\kappa_n}{n-1} \frac{M(r)}{A_{n-1}} \times \frac{1}{r^{n-2}}\right]^{-1}$$ ✓

c) $\nabla^2(1 - e^{-\lambda}) = \nabla^2\left(\frac{2\kappa_n}{n-1} \frac{1}{r^{n-2}} \frac{M(r)}{A_{n-1}}\right) = \frac{2\kappa_n}{n-1} \frac{1}{r^{n-1}} \frac{d}{dr}\left(r^{n-1} \frac{d}{dr}\left[\frac{1}{r^{n-2}} \frac{M(r)}{A_{n-1}}\right]\right)$

$= \frac{2\kappa_n}{n-1} \frac{1}{r^{n-1}} \frac{d}{dr}\left[r^{n-1}\left((2-n) r^{1-n} \frac{M(r)}{A_{n-1}} + \frac{1}{r^{n-2}} \underbrace{\frac{d}{dr}\int_0^r dr'\, r'^{n-1} \rho(r')}_{r^{n-1}\rho(r)}\right)\right] = \frac{2\kappa_n}{n-1} \frac{1}{r^{n-1}} \frac{d}{dr}\left[(2-n)\int_0^r dr'\, r'^{n-1} \rho(r') + r^n \rho(r)\right]$

$= \frac{2\kappa_n}{n-1}\left[(2-n)\rho(r) + \frac{1}{r^{n-1}} \frac{d}{dr}\left(r^{n-1} \underbrace{(\rho(r)\, r)}\right)\right] \Rightarrow \nabla^2(1 - e^{-\lambda}) = \frac{2\kappa_n}{n-1}\left[(2-n)\rho + \vec{\nabla}\cdot(\rho \vec{x})\right]$ ✓

↳ The r^{th} component of $\rho(r) \vec{x}$ (vector in (n+1) dimension)

$$\Rightarrow \nabla^2(1 - e^{-\lambda}) = \frac{2\kappa_n}{n-1}\left[(2-n)\rho(r) + n\rho(r) + r\frac{d\rho}{dr}\right] = \frac{2\kappa_n}{n-1} \frac{(r^2 \rho(r))'}{r}$$ ✓

d) $\left.\begin{array}{l} G^0{}_0 = -\kappa_n \rho(r) \\ G^1{}_1 = \kappa_n p(r) \end{array}\right\} \Rightarrow G^1{}_1 - G^0{}_0 = \kappa_n(\rho + p) \Rightarrow \frac{n-1}{2r} e^{-\lambda}(\nu' + \lambda') = \kappa_n(\rho + p)$

$p \ll \rho$, $\nu, \lambda \ll 1 \overset{(18)}{\Rightarrow} \frac{n-1}{2r}(\nu' + \lambda') = \kappa_n \rho$. Weak field limit in (15) : $\lambda \simeq \frac{2\kappa_n}{n-1} \frac{M(r)}{A_{n-1}\, r^{n-2}} = \frac{2\kappa_n}{n-1} \frac{1}{r^{n-2}} \int_0^r dr'\, r'^{n-1} \rho(r')$

$\Rightarrow \lambda' = \frac{2\kappa_n}{n-1}\left[(2-n) r^{1-n} \frac{M(r)}{A_{n-1}} + r\rho\right] \longrightarrow \nu' = \frac{2(n-2)}{n-1} \kappa_n r^{1-n} \frac{M(r)}{A_{n-1}} \Rightarrow \nu'' = \frac{2(n-2)}{n-1} \kappa_n \left[(1-n) r^{-n} \frac{M(r)}{A_{n-1}} + \rho\right]$

$$\Rightarrow \nabla^2 \nu = (n-1)\frac{\nu'}{r} + \nu'' = 2\,\frac{n-2}{n-1} \kappa_n \rho$$ ✓