Problem Set 10

due Monday 23rd of Azar in TA class

1- We found in the class that in the weak field limit $g_{00} = -1 - 2\phi$. This is true only in four spacetime dimensions. Show that the non-relativistic weak field limit of Einstein's equations in an n-dimensional space ($n+1$ spacetime dimensions) reads

$$\nabla^2 g_{00} = -2\frac{n-2}{n-1}\kappa_n \rho, \tag{1}$$

where ∇^2 is the Laplacian on the *flat* $\mathbb{R}^n$, and κ_n is the strength of gravity that appears in $G_{\mu\nu} = \kappa_n T_{\mu\nu}$ in n dimensions.[1] [Hint: Follow the steps of §12.10 of d'Inverno starting from Eq. (12.67) (valid for any signature and in any dimension),

$$\frac{1}{2}\epsilon\Box\left(h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu}\eta^{\rho\sigma}h_{\rho\sigma}\right) = -\kappa_n T_{\mu\nu}, \tag{2}$$

for $\epsilon h_{\mu\nu} = g_{\mu\nu} - \eta_{\mu\nu}$ where $\Box = \eta^{\mu\nu}\partial_\mu\partial_\nu$.]

2- Poisson's equation in an n-dimensional space can be written as

$$\nabla^2\phi = G_n^\Phi \rho. \tag{3}$$

For example, in 3 dimensions G_3^Φ is 4π times the usual Newton's constant. To better understand Poisson's equation and the meaning of G_n^Φ, we may consider its integral form $\int \nabla\phi\cdot\hat{\mathbf{n}}dA = G_n^\Phi M$, where the integral is over an $(n-1)$-dimensional surface A surrounding a mass M. This means that G_n^Φ (the "flux-based" Newton's constant) measures the flux of gravitational field lines per unit enclosed mass.
(a) Show that Eqs. (1) and (3) can be related by

$$g_{00} = -1 - 2\frac{n-2}{n-1}\frac{\kappa_n}{G_n^\Phi}\phi. \tag{4}$$

(b) Show that the Newtonian gravitational potential of a spherically symmetric mass distribution is

$$\phi = \frac{G_n^\Phi}{A_{n-1}}\int_\infty^r \frac{M(r')}{r'^{n-1}}dr', \tag{5}$$

where

$$M(r) = A_{n-1}\int_0^r \rho(r')r'^{n-1}dr' = \int_0^r \rho(\mathbf{x})d^n\mathbf{x} \tag{6}$$

is the mass below radius r, and

$$A_{n-1} = \frac{2\pi^{n/2}}{\Gamma(n/2)} \tag{7}$$

is the area of an $(n-1)$-dimensional unit sphere in $\mathbb{R}^n$.
(c) Show that outside the sphere this reduces to

$$\phi = -\frac{G_n^\Phi M}{(n-2)A_{n-1}} \times \frac{1}{r^{n-2}}, \tag{8}$$

where M is the total mass. In analogy to Newton's original formula for G, we can define a "radius-based" Newton's constant G_n^R by

$$\phi = -\frac{G_n^R M}{r^{n-2}}. \tag{9}$$

Conclude that

$$G_n^\Phi = (n-2)A_{n-1}G_n^R. \tag{10}$$

3- Consider a static spherically symmetric metric in $n+1$ spacetime dimensions:

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\lambda(r)}dr^2 + r^2\left(d\theta_1^2 + \sin^2\theta_1 d\theta_2^2 + \cdots + \sin^2\theta_1\cdots\sin^2\theta_{n-2}d\theta_{n-1}^2\right). \tag{11}$$

[1] Equivalently, κ_n appears in the action as $S = -\frac{1}{2\kappa_n}\int\sqrt{-g}Rd^{n+1}x + S_m$. What is the unit of κ_n? The "reduced Planck mass" in n spatial dimensions is related to κ_n via $\kappa_n = M_n^{1-n}$.

It can be shown that the Einstein tensor has a tt component

$$G^0{}_0 = \frac{(n-1)(n-2)}{2r^2}\left(e^{-\lambda}-1\right) - \frac{n-1}{2r}\lambda' e^{-\lambda} \tag{12}$$

$$= \frac{n-1}{2r^{n-1}}\frac{d}{dr}\left[r^{n-2}(e^{-\lambda}-1)\right] \tag{13}$$

(compare with Eq. (14.36) of d'Inverno).

(a) Assume that we have a perfect fluid with energy momentum tensor[2]

$$T^{\mu\nu} = (\rho(r)+p(r))u^\mu u^\nu + p(r)g^{\mu\nu}. \tag{14}$$

Show that since the fluid is static and spherically symmetric, like the metric, then $u^\mu = (e^{-\nu/2}, 0, 0, \cdots)$ and hence $T^0{}_0 = -\rho$.

(b) Deduce that $G^0{}_0 = \kappa_n T^0{}_0$ implies that the solution with finite $\lambda(0)$ is

$$e^{\lambda(r)} = \left[1 - \frac{2\kappa_n M(r)}{(n-1)A_{n-1}} \times \frac{1}{r^{n-2}}\right]^{-1}, \tag{15}$$

where $M(r)$ and A_n are given by Eqs. (6) and (7) (note that the integral in Eq. (6) is over the *flat* $\mathbb{R}^n$).

(c) Show that

$$\nabla^2\left(1-e^{-\lambda}\right) = \frac{2\kappa_n}{n-1}\frac{(r^2\rho)'}{r} = \frac{2\kappa_n}{n-1}\left[(2-n)\rho + \nabla\cdot(\rho\mathbf{x})\right], \tag{16}$$

where $\nabla^2 = r^{1-n}\frac{d}{dr}(r^{n-1}\frac{d}{dr})$ is the Laplacian on the flat $\mathbb{R}^n$. This means that when $\rho = 0$, we have to solve Laplace's equation—just like the case of Newtonian gravity. It is easy to show directly that Eq. (20) solves Eq. (16) both inside and outside the spherical mass distribution.

(d) It may seem a little strange at first sight that in Eq. (20) $1-e^{-\lambda}$ is proportional to $M(r)/r^{n-2}$, whereas the ϕ in Eq. (8) is proportional to $\int M(r)dr/r^{n-1}$. Also, one might have expected that Eq. (16) would read $\nabla^2\left(1-e^{-\lambda}\right) \propto \rho$, which it doesn't. But you should remember that the weak field limit of Einstein's equations connects the tt (not rr) component of the metric to the Newtonian potential (see Eq. (4)). In other words, it's $\nu(r)$, not $\lambda(r)$, that must be related to ϕ. Let us see if it really is.

The rr component of Einstein's tensor for the metric (11) is given by

$$G^1{}_1 = \frac{(n-1)(n-2)}{2r^2}\left(e^{-\lambda}-1\right) + \frac{n-1}{2r}\nu' e^{-\lambda} \tag{17}$$

(compare with Eq. (14.38) of d'Inverno). Noting that $T^1{}_1 = p$, show that $G^1{}_1 = \kappa_n T^1{}_1$ implies

$$\frac{n-1}{2r}(\nu'+\lambda')e^{-\lambda} = \kappa_n(\rho+p). \tag{18}$$

Conclude that in the non-relativistic $p \ll \rho$ weak field limit $\lambda, \nu \ll 1$ we have

$$\nabla^2\nu = 2\frac{n-2}{n-1}\kappa_n\rho, \tag{19}$$

in complete agreement with the weak field limit Eq. (1). Thus in this limit

$$e^{\nu(r)} = 1 + 2\frac{n-2}{n-1}\frac{\kappa_n}{A_{n-1}}\int_\infty^r \frac{M(r')}{r'^{n-1}}dr'. \tag{20}$$

So in general $\nu \neq -\lambda$, but outside the mass distribution, $\nu = -\lambda$.

[2] A non-vacuum static solution is impossible without pressure support.