440
Differential Equations

Problem Set 9

due Tuesday 22nd of Ordibehesht in TA class

1- [Simmons §58-6] Find the critical points of

(a)

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - (x^3 + x^2 - 2x) = 0; \tag{1}$$

(b)

$$\begin{cases} \dfrac{dx}{dt} = y^2 - 5x + 6 \\ \dfrac{dy}{dt} = x - y. \end{cases} \tag{2}$$

***2-** [Simmons §58-7] Find all solutions of the nonautonomous system

$$\begin{cases} \dfrac{dx}{dt} = x \\ \dfrac{dy}{dt} = x + e^t. \end{cases} \tag{3}$$

and sketch (in the xy-plane) some of the curves defined by these solutions. [The take-away lesson from this problem is that for non-autonomous systems, the paths may intersect each other, so the phase space can become very messy and of little use.]

3- [Simmons §60-1] Determine the nature and stability properties of the critical point $(0,0)$ for each of the following linear autonomous systems:

(a)

$$\begin{cases} \dfrac{dx}{dt} = 2x \\ \dfrac{dy}{dt} = 3y; \end{cases} \tag{4}$$

(b)

$$\begin{cases} \dfrac{dx}{dt} = -x - 2y \\ \dfrac{dy}{dt} = 4x - 5y; \end{cases} \tag{5}$$

(c)

$$\begin{cases} \dfrac{dx}{dt} = -3x + 4y \\ \dfrac{dy}{dt} = -2x + 3y; \end{cases} \tag{6}$$

(d)

$$\begin{cases} \dfrac{dx}{dt} = 5x + 2y \\ \dfrac{dy}{dt} = -17x - 5y; \end{cases} \tag{7}$$

(e)

$$\begin{cases} \dfrac{dx}{dt} = -4x - y \\ \dfrac{dy}{dt} = x - 2y; \end{cases} \tag{8}$$

(f)

$$\begin{cases} \dfrac{dx}{dt} = 4x - 3y \\ \dfrac{dy}{dt} = 8x - 6y; \end{cases} \tag{9}$$

(g)

$$\begin{cases} \dfrac{dx}{dt} = 4x - 2y \\ \dfrac{dy}{dt} = 5x + 2y. \end{cases} \tag{10}$$

4- [Simmons §60-3] **(a)** If $a_1 b_2 - a_2 b_1 \neq 0$, show that the system

$$\begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y + c_1 \\ \dfrac{dy}{dt} = a_2 x + b_2 y + c_2 \end{cases} \tag{11}$$

has a single isolated critical point (x_0, y_0).
(b) Show that the system in (a) can be written in the form of

$$\begin{cases} \dfrac{dx}{dt} = a_1 x + b_1 y \\ \dfrac{dy}{dt} = a_2 x + b_2 y \end{cases} \tag{12}$$

by means of the change of variables $\bar{x} = x - x_0$ and $\bar{y} = y - y_0$.
(c) Find the critical point of the system

$$\begin{cases} \dfrac{dx}{dt} = 2x - 2y + 10 \\ \dfrac{dy}{dt} = 11x - 8y + 49, \end{cases} \tag{13}$$

write the system in the form of (12) by changing the variables, and determine the nature and stability properties of the critical point.

5- [Simmons §60-4] In Section 20 we studied the free vibrations of a mass attached to a spring by solving the equation

$$\frac{d^2x}{dt^2} + 2b\frac{dx}{dt} + a^2 x = 0, \tag{14}$$

where $b \geq 0$ and $a > 0$ are constants representing the viscosity of the medium and the stiffness of the spring, respectively. Consider the equivalent autonomous system

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -a^2 x - 2by, \end{cases} \tag{15}$$

which has $(0, 0)$ as its only critical point.
(a) Find the auxiliary equation of (15). What are p and q? [Recall that $-p$ is defined as the sum of the eigenvalues and q is defined as their product.]
(b) For each of the following four cases, describe the nature and stability properties of the critical point, and give a brief physical interpretation of the corresponding motion of the mass: **(i)** $b = 0$; **(ii)** $0 < b < a$; **(iii)** $b = a$; **(iv)** $b > a$.