440
Differential Equations

Problem Set 12

due Tuesday 12th of Khordad in TA class

1- The PDE

$$\frac{\partial f}{\partial t} = a^2 \frac{\partial^2 f}{\partial x^2} \tag{1}$$

is called the "Heat equation". a is a constant and the physical interpretation of $f(x,t)$ is the temperature of a rod of length L at position x and time t. Use the method of separation of variables to solve this equation with the boundary condition

$$f(0,t) = f(L,t) = 0, \quad \forall t. \tag{2}$$

[You can find the solution in §41 of Simmons.]

2- The two-dimensional Laplace's equation in polar coordinates reads:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial f}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 f}{\partial \theta^2} = 0. \tag{3}$$

Use the method of separation of variables to solve it, noting the continuity of $f(r,\theta)$, i.e.,

$$f(r,\theta) = f(r,\theta+2\pi), \quad \forall r,\theta. \tag{4}$$

[You can find the solution in §42 of Simmons.]

***3-** The two-dimensional Laplace's equation in Cartesian coordinates reads:

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \tag{5}$$

We want to find solutions in the semi-infinite region $0 \le x \le L$, $0 \le y$. Use the method of separation of variables to solve it with the boundary conditions

$$f(0,y) = f(L,y) = 0, \quad \forall y; \qquad \text{and} \qquad \lim_{y\to+\infty} f(x,y) = 0. \tag{6}$$

4- [Simmons §25-3] Find the normal form of Bessel's equation

$$x^2y'' + xy' + (x^2 - p^2)y = 0, \tag{7}$$

and use it to show that every nontrivial solution has infinitely many positive zeros.

***5-** [Simmons §25-Theorem C] [Prove:] Let $y_p(x)$ be a nontrivial solution of Bessel's equation on the positive x-axis. If $0 < p < 1/2$, then every interval of length π contains at least one zero of $y_p(x)$; if $p = 1/2$, then the distance between successive zeros of $y_p(x)$ is exactly π; and if $p > 1/2$, then every interval of length π contains at most one zero of $y_p(x)$.

6- Show that if $|a| \gg |b|$ and $a^2 \gg |b\alpha|$, then the equation

$$\frac{d^2x}{dt^2} + (a + b\sin(\alpha t))^2 x = 0 \tag{8}$$

can be solved approximately by the WKB method. Find this approximate solution.