Problem Set 10

due Tuesday 29th of Ordibehesht in TA class

1- Consider the system

$$\begin{cases} \dfrac{dx}{dt} = F(x,y) \\ \dfrac{dy}{dt} = G(x,y). \end{cases} \tag{1}$$

Suppose that $F(x_0, y_0) = G(x_0, y_0) = 0$, so that (x_0, y_0) is a critical point. Taylor expand F and G around the critical point and show that the corresponding linearized system is

$$\begin{cases} \dfrac{d\bar{x}}{dt} = \left.\dfrac{\partial F}{\partial x}\right|_0 \bar{x} + \left.\dfrac{\partial F}{\partial y}\right|_0 \bar{y} \\ \dfrac{d\bar{y}}{dt} = \left.\dfrac{\partial G}{\partial x}\right|_0 \bar{x} + \left.\dfrac{\partial G}{\partial y}\right|_0 \bar{y}, \end{cases} \tag{2}$$

where $\bar{x} = x - x_0$, $\bar{y} = y - y_0$, and derivatives are evaluated at $(\bar{x} = 0, \bar{y} = 0)$. Show that the critical point is isolated, if the following matrix has nonzero determinant:

$$J = \left.\begin{pmatrix} \dfrac{\partial F}{\partial x} & \dfrac{\partial F}{\partial y} \\ \dfrac{\partial G}{\partial x} & \dfrac{\partial G}{\partial y} \end{pmatrix}\right|_{\bar{x}=0,\bar{y}=0}. \tag{3}$$

Note that the eigenvalues of J determine the stability properties of the system, too.

2- The system

$$\begin{cases} \dfrac{dx}{dt} = 14x - \dfrac{1}{2}x^2 - xy \\ \dfrac{dy}{dt} = 16y - \dfrac{1}{2}y^2 - xy \end{cases} \tag{4}$$

is a special case of the "competing species" problem. Draw its phase portrait by linearizing it around its critical points (you can use the results of problem 1). [You can find a detailed solution on MIT OpenCourseWare. You can also enter `StreamPlot[{14*x-x^2/2-x*y,16*y-y^2/2-x*y},{x,0,35},{y,0,35}]` at www.wolframalpha.com to visualize the phase portrait.]

3- Draw the phase portrait of the following systems by linearizing them around their critical points (you can use the results of problem 1):

(a)

$$\begin{cases} \dfrac{dx}{dt} = y \\ \dfrac{dy}{dt} = -x + x^3. \end{cases} \tag{5}$$

[You'll find that $(0,0)$ is a center of the linearized system. I assure you that it remains a center even for the nonlinear system.]

***(b)**

$$\begin{cases} \dfrac{dx}{dt} = y^2 - 3x + 2 \\ \dfrac{dy}{dt} = x^2 - y^2. \end{cases} \tag{6}$$

***4-** The system

$$\begin{cases} \dfrac{dx}{dt} = 2xy \\ \dfrac{dy}{dt} = y^2 - x^2 \end{cases} \tag{7}$$

has a critical point which is not simple (if you compute the matrix J of problem 1, then $\det J = 0$). Find the paths of the system by solving $dy/dx = (y^2 - x^2)/2xy$ (change the variable to $z = y/x$) and show that they are $(x-c)^2 + y^2 = c^2$. Depict the phase portrait.

In the rest of the problems you should use potential energy and analogy with mechanical systems, as explained in Section 63 of Simmons.

5- [Simmons §63-1] If $f(0) = 0$ and $xf(x) > 0$ for $x \neq 0$, show that the paths of

$$\frac{d^2x}{dt^2} + f(x) = 0 \tag{8}$$

are closed curves surrounding the origin in the phase plane; that is, show that the critical point $x = 0$, $y = dx/dt = 0$ is a stable but not asymptotically stable center. Describe this critical point with respect to its nature and stability if $f(0) = 0$ and $xf(x) < 0$ for $x \neq 0$.

6- [Simmons §63-2] Most actual springs are not linear. A nonlinear spring is called hard or soft according as the magnitude of the restoring force increases more rapidly or less rapidly than a linear function of the displacement. The equation

$$\frac{d^2x}{dt^2} + kx + \alpha x^3 = 0, \qquad k > 0, \tag{9}$$

describes the motion of a hard spring if $\alpha > 0$ and a soft spring if $\alpha < 0$. Sketch the paths in each case. [Problem 3(a) corresponds to $k = 1$ and $\alpha = -1$, which you previously solved by linearization.]

7- [Simmons §63-3] Find the equation of the paths of

$$\frac{d^2x}{dt^2} - x + 2x^3 = 0, \tag{10}$$

and sketch these paths in the phase plane. Locate the critical points and determine the nature of each.