Problem Set 12

due on the final exam

1- [optional, not to be graded; Rindler 7.7—using non-rationalized units] Although it is natural to pair the four Maxwell equations as in (7.26) and (7.27), each pair corresponding to one of the 4-tensor equations (7.40), (7.41), yet it is also instructive to pair div $\mathbf{e} = 4\pi\rho$ with div $\mathbf{b} = 0$. These are the two equations that contain no time derivatives, and which can, in fact, be regarded as 'constraints' on the initial conditions. Verify that they are 'propagated' by the other two equations (the 'evolution equations') plus the equation of continuity (7.37). In other words, if initial conditions (fields and sources) are prescribed on a surface $t = t_0$, satisfying the constraints, then if we use the evolution equations to calculate the future, this will automatically continue to satisfy the constraints. [Hint: Consider, for a start, $(\partial/\partial t)(\nabla \cdot \mathbf{e} - 4\pi\rho)$ and use (7.26)(ii) and (7.37).]

2- [optional, not to be graded] a) Let F be the matrix representation of a rank-$(1,1)$ tensor $F^{\bullet}{}_{\bullet}$. Prove that the eigenvalues of F are invariant under Lorentz transformations.

b) Show that for the electromagnetic field strength tensor, the eigenvalues satisfy the equation

$$\lambda^4 - (E^2 - B^2)\lambda^2 - (\mathbf{E} \cdot \mathbf{B})^2 = 0. \tag{1}$$

c) Again for the electromagnetic field strength tensor, show that

$$\operatorname{tr} F^2 = 2(E^2 - B^2), \qquad \operatorname{tr} F^4 = 2(E^4 + B^4) - 4(\mathbf{E} \times \mathbf{B})^2 = 2(E^2 - B^2)^2 + 4(\mathbf{E} \cdot \mathbf{B})^2, \qquad \det F = -(\mathbf{E} \cdot \mathbf{B})^2, \tag{2}$$

while $\operatorname{tr} F = \operatorname{tr} F^3 = 0$. [Hint: You may use computer software, but it's still wise to go to a coordinate where $\mathbf{E}$ lies along the z-direction and $\mathbf{B}$ is in the x-z plane.]

3- Show that the electromagnetic energy-momentum tensor has vanishing trace: $T^{\mu}{}_{\mu} = 0$.

4- Recall the components of the energy-momentum tensor: $T^{00} = u = \frac{1}{2}(E^2 + B^2)$ (the energy density) and $T^{0i} = S_i = (\mathbf{E} \times \mathbf{B})_i$ (Poynting vector). Prove that

$$u^2 - S^2 = \frac{1}{16}\left[(F_{\mu\nu}F^{\mu\nu})^2 + \left(F_{\mu\nu}\tilde{F}^{\mu\nu}\right)^2\right]. \tag{3}$$

5- Compute the time averaged energy-momentum tensor for a plane wave with wave vector $\mathbf{k}$ and linear polarization vector $\boldsymbol{\epsilon}$ (i.e., $\mathbf{E} \parallel \boldsymbol{\epsilon} \perp \mathbf{k}$ and $|\boldsymbol{\epsilon}| = 1$). Then average over all polarization vectors $\boldsymbol{\epsilon}$ that are perpendicular to $\mathbf{k}$. Finally average over all directions of propagation.

6- We saw in the class that a term $q\int A_\mu U^\mu d\tau$ in the action of a particle can describe its interaction with the electromagnetic field. Similarly a term $\int A_\mu J^\mu d^4x$ describes the interaction of a current with the field. Is this interaction gauge-invariant? [Hint: write $A_\mu \to A_\mu + \partial_\mu f$ and perform an integration by parts, discarding the surface terms.]

7- a) In the class we found that the action

$$S[x^i] = -m_0 \int d\tau = -m_0 \int \sqrt{1 - \sum_{i=1}^{3}\left(\frac{dx^i}{dt}\right)^2}\, dt \tag{4}$$

describes the motion of a free particle. There are three functions $x^1(t)$, $x^2(t)$, and $x^3(t)$ which appear in this action. Now consider

$$S[x^\mu] = -m_0 \int d\tau = -m_0 \int \sqrt{\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}}\, d\tau, \tag{5}$$

which has four functions $x^1(\tau)$, $x^2(\tau)$, $x^3(\tau)$, and $x^0(\tau)$. Show that this action describes the motion of a free particle too.

b) Find the equations of motion for the action

$$S[e, x^\mu] = -\frac{1}{2}\int \left[e^{-1}\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + m_0^2 e\right] d\tau, \tag{6}$$

which involves five functions of τ, and show that they, too, describe the motion of a free particle of mass m_0 (which can be zero).