Mathematical Physics

Problem Set 8

due 1pm on Saturday 14th of Day in my office

1- [Byron-Fuller 5.10 with modified notation] Maxwell's equations in free space in SI units are:

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{B} = \epsilon_0 \mu_0 \frac{\partial \mathbf{E}}{\partial t}, \qquad \nabla \cdot \mathbf{E} = 0. \tag{1}$$

At any time t the electric (and magnetic) fields may be Fourier analyzed:

$$\mathbf{E}(\mathbf{r}, t) = \frac{1}{(2\pi)^{3/2}} \int_{-\infty}^{\infty} \mathbf{E}(\mathbf{k}, t) e^{i\mathbf{k}\cdot\mathbf{r}} d\mathbf{k}, \qquad \text{etc.} \tag{2}$$

[Here the Fourier components of $\mathbf{E}(\mathbf{r}, t)$ have been written $\mathbf{E}(\mathbf{k}, t)$; the argument identifies $\mathbf{E}(\mathbf{k}, t)$ as the Fourier component of the field, distinguishing it from the field itself, which has argument $\mathbf{r}$. If you don't like this notation, adopt another.]

a) Prove that the Fourier components of the electric and magnetic fields satisfy the following equations:

i) $\dot{\mathbf{B}}(\mathbf{k}, t) = -i\mathbf{k} \times \mathbf{E}(\mathbf{k}, t)$

ii) $\mathbf{k} \cdot \mathbf{B}(\mathbf{k}, t) = 0$

iii) $\dot{\mathbf{E}}(\mathbf{k}, t) = ic^2 \mathbf{k} \times \mathbf{B}(\mathbf{k}, t)$

iv) $\mathbf{k} \cdot \mathbf{E}(\mathbf{k}, t) = 0$.

These four equations are equivalent to Maxwell's equations. This reformulation is well-suited as a starting point for quantum mechanical considerations of the electromagnetic field. [use $\epsilon_0 \mu_0 = c^{-2}$.]

b) Using the results of part (a), prove that the Fourier components of the electric field satisfy the homogeneous wave equation. That is, prove

$$\left[\frac{\partial^2}{\partial t^2} + c^2 k^2\right] \mathbf{E}(\mathbf{k}, t) = 0. \tag{3}$$

2- [Byron-Fuller 5.20] Let $\{Q_n\}$ be an orthonormal system of Sturm-Liouville polynomials. Prove that the functions Q'_1, Q'_2,... form an orthogonal system of polynomials with weight function $w\alpha$, where the notation is that of Section 5.10. [Hint: For Sturm-Liouville polynomial, $\gamma = 0$.]

3- [Byron-Fuller 5.23] Consider Bessel's equation $Bf_n = n^2 f_n$ on the interval $[0, \infty]$, where

$$B = x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} + x^2. \tag{4}$$

a) Using the methods discussed in Section 5.10, find the weight function appropriate to Bessel's equation.

[...]

4- Laguerre polynomials are solutions of the differential equation

$$xy''(x) - [x - (s+1)]y'(x) + ny(x) = 0 \tag{5}$$

on the interval $[0, \infty]$. Here $s > -1$ is a fixed number but n can take any non-negative integer value. Find the weight function that should be used in their inner product to maintain orthogonality.

5- Using the results of Sturm-Liouville theory and the equation

$$y''(x) + n^2 y(x) = 0, \tag{6}$$

prove that $\sin nx$ and $\cos nx$ are orthogonal functions that can be used as a basis for periodic[1] square-integrable functions on $[-\pi, \pi]$.

[1] i.e., $f(-\pi) = f(\pi)$ *and* $f'(-\pi) = f'(\pi)$.