Mathematical Physics

Problem Set 7

due Tuesday 26th of Azar in the TA class

1- [Byron-Fuller 5.1] Prove that $\int_{-1}^{1} P_n(x)dx = 0$ for $n \neq 0$ [P_n is Legendre polynomial of order n ...].

2- a) Is Dirac δ function even or odd, or neither?
b) By a change of the integration variable in $\int_{-\infty}^{\infty} f(x)\delta(x)dx = f(0)$ from x to $y = x/a$ show that $\delta(ax) = |a|^{-1}\delta(x)$.
c) [optional; not to be graded] Let $f(x)$ be an arbitrary function of x with only one simple root at x_0 (i.e., x_0 is not a multiple root; in other words, $f'(x_0) \neq 0$). Prove that $\delta(f(x)) = |f'(x_0)|^{-1}\delta(x - x_0)$.

3- Consider the "square wave" function defined on $[-1, 1]$:

$$f(x) = \begin{cases} 0 & 1 > x > \frac{1}{2} \\ 1 & \frac{1}{2} \geq x \geq -\frac{1}{2} \\ 0 & -\frac{1}{2} > x > -1 \end{cases} \tag{1}$$

a) Compute the coefficients f_0, f_1, and f_2 in the expansion of f in terms of Legendre polynomials.
b) Compute the coefficients f_n in the Fourier expansion of f. Could you guess what terms would be absent?
c) Use Parseval's theorem to find a closed form answer for $\sum |f_n|^2$ of part (b).

4- Find the Fourier series of the function $f(x) = x^2$ on the interval $[-\pi, \pi]$.

5- What is the Fourier transformation of Dirac δ function?

6- a) Let $f(x)$ be a square-integrable function and let $g(x) = f'(x)$ be its derivative. Denote the Fourier transform of $f(x)$ by $\tilde{f}(k)$ and that of $g(x)$ by $\tilde{g}(k)$. Show that $\tilde{g}(k) = ik\tilde{f}(k)$. [See Eq. (5.69) in Byron-Fuller.]
b) Use part (a) to obtain the Fourier transformation of the so-called "error function"

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt. \tag{2}$$

7- [optional, not to be graded] Solve the ordinary differential equation $f''(x) + k^2 f(x) = 0$ by substituting the Fourier integral of f.