Mathematical Physics

Problem Set 6

due Tuesday 19th of Azar in the TA class

1- [Byron-Fuller 3.16] Show that the eigenvalues of

$$M = \begin{bmatrix} 3 & 5 & 8 \\ -6 & -10 & -16 \\ 4 & 7 & 11 \end{bmatrix} \tag{1}$$

are $\lambda = 0, 1, 3$, and that the corresponding eigenvectors are

$$\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 4 \\ -8 \\ 5 \end{bmatrix}. \tag{2}$$

Construct a diagonalizing matrix P, prove that its inverse exists, compute its inverse, and verify that $P^{-1}MP$ is diagonal with the eigenvalues on the diagonal. Note that $\det M = 0$. Is it true that any diagonalizable matrix with an eigenvalue equal to zero is singular?

2- [Byron-Fuller 3.17] Denote the two-dimensional rotation matrix for a rotation of the coordinate axes through an angle θ by

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{3}$$

Find the eigenvalues and eigenvectors of A. Find a diagonalizing matrix for A, that is, a matrix P such that $P^{-1}AP$ is a diagonal matrix. Demonstrate that P is such a matrix by inverting P and forming the product $P^{-1}AP$.

3- [Byron-Fuller 3.18] Show that if

$$A = \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \tag{4}$$

[with $a \neq 0$] there exists no nonsingular matrix P of order 2 such that $P^{-1}AP = D$, where D is a diagonal matrix. This example shows that not every matrix can be diagonalized by a similarity transformation.

4- [similar to Byron-Fuller 3.21] Assume that the $n \times n$ matrix A is nonsingular and has n distinct eigenvalues. Show that its inverse is given by

$$A^{-1} = \sum_{i=1}^{n} \lambda_i^{-1} v_i v_i^T \tag{5}$$

where v_i is the eigenvector corresponding to the eigenvalue λ_i.

5- [Byron-Fuller 3.22] Consider an $n \times n$ matrix Q whose elements are given by $Q_{ij} = v_i v_j$, where v is a vector in n-dimensional space. Find the eigenvalues and eigenvectors of Q. Are the eigenvectors all unique? [Perhaps it'll be simpler if you choose the coordinate axes such that v lies on the x_1 axis. Incidentally, note that in this problem v_i means the i-th component of the vector v — a departure from the standard convention of Byron-Fuller.]

6- In the class we proved that if A is a self-adjoint operator then its matrix representation in an *orthonormal* basis satisfies $[A]^*_{ij} = [A]_{ji}$ (the unnumbered equation before Theorem 4.9 in Byron-Fuller). Is this true when the basis is not orthonormal? Either prove or give a counterexample.

7- [Byron-Fuller 4.17] The *projection* operator P_i is defined as follows:

$$x' = P_i x \equiv x_i(x_i, x), \tag{6}$$

where x_i is a unit vector. This operator is called a projection operator because all x' are in the direction of x_i and the length of x' equals the component of x in the x_i direction, namely (x_i, x). In the following, let the x_i be a set of orthonormal vectors which span the n-dimensional vector space V. Prove:

a) P_i is idempotent [i.e., $P_i^2 = P_i$].

b) $P_i P_j = 0$ for $i \neq j$. Interpret geometrically.

c) P_i has no inverse.

d) $\sum_{i=1}^{n} P_i = I$.

e) P_i is Hermitian.

f) [optional, not to be graded] Let A be a self-adjoint linear operator defined on an n-dimensional vector space V with n eigenvalues, ϵ_i, and n eigenvectors, x_i. Prove that A may be written as

$$A = \sum_{i=1}^{n} \epsilon_i P_i, \tag{7}$$

where P_i is as defined above. This is known as the spectral theorem for self-adjoint linear operators. (The analogous theorem for normal operators can also be easily proved.) Note the consistency of this result with parts (d) and (e) of this problem.

8- [optional, not to be graded; Byron-Fuller 4.6] Either prove or find a counterexample to the following statements:

a) If A and B are $n \times n$ matrices, then $AB = 0$ implies that either $A = 0$ or $B = 0$.

b) An $n \times n$ Hermitian matrix which is completely degenerate (all eigenvalues equal) is necessarily diagonal.

c) If an $n \times n$ matrix is Hermitian, then all its powers are Hermitian.

d) $\det(A + B) = \det A + \det B$.

e) If A and B are $n \times n$ Hermitian matrices, then AB is Hermitian.

f) $\mathrm{tr}AB = \mathrm{tr}A\mathrm{tr}B$ in a finite-dimensional space.

9- [optional, not to be graded; Byron-Fuller 4.4] Let B be an $n \times n$ matrix. Define the $n \times n$ matrix $A = e^{iB}$ in terms of the power-series expansion of the exponential. That is,

$$A = \sum_{n=0}^{\infty} \frac{(iB)^n}{n!} = I + iB + \frac{(iB)^2}{2!} + \cdots \tag{8}$$

Assuming that the series converges, prove that A is isometric if B is self-adjoint. Also show that any isometry U can be written as $U = \exp(iH)$, where H is Hermitian. Obtain an expression for H in terms of quantities related to U. [Hint: the properties of the eigenvectors and eigenvalues of the operators will be helpful.] [In finite-dimensional spaces like this an isometry (isometric matrix) is just the same thing as a unitary matrix, which I defined in the class.]

10- [optional, not to be graded] Let $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

a) Compute e^A directly from the series (8) (without the i).

b) Find a similarity transformation that diagonalizes A. In the new basis the same linear operator is represented by the diagonal matrix $D = P^{-1}AP$; so it's easy to exponentiate D: just exponentiate each diagonal element. Now switch back to the original basis and obtain $e^A = Pe^D P^{-1}$. Compare with part (a).