Mathematical Physics

Problem Set 5

due Tuesday 5th of Azar in the TA class

1- Consider a homogeneous 2nd order linear differential equation

$$a(x)y''(x) + b(x)y'(x) + c(x)y(x) = 0, \tag{1}$$

where a, b, and c are given functions of x. Let V be the set of all real solutions $y(x)$ of this equation. Prove that V is a vector space and show that it's 2-dimensional and isomorphic to $\mathbb{R}^2$.

2- Let $X = \{x_1, x_2\}$ be a basis for $\mathbb{R}^2$. Think about x_i as the usual unit vectors $\hat{\mathbf{e}}_i$.
a) A linear operator A is defined as follows:

$$Ax_1 = \cos\theta x_1 + \sin\theta x_2, \qquad Ax_2 = -\sin\theta x_1 + \cos\theta x_2. \tag{2}$$

Find the matrix associated to A in the X basis. Geometrically what does A do to a vector?
b) Another linear operator B is defined as follows:

$$Bx_1 = x_1, \qquad Bx_2 = 0. \tag{3}$$

Find the matrix associated to B in the X basis. Geometrically what does B do to a vector?
c) A third linear transformation S is defined as follows:

$$Sx_1 = x_1 + x_2, \qquad Sx_2 = -x_2. \tag{4}$$

S is used to define a new basis $Y = \{y_1 = Sx_1, y_2 = Sx_2\}$. Draw both bases X and Y on the same diagram.
d) x is a vector with coordinate (α_1, α_2) in the X basis. Find the coordinates of x in the Y basis. Use matrix multiplication.
e) Find the matrix representation of A and B in the Y basis. Use matrix multiplication a la (3.35) in Byron-Fuller.

3- [Byron-Fuller 3.9] Define the commutator $[A, B]$ of two matrices by $[A, B] = AB - BA$. Show that
a) $[AB, C] = A[B, C] + [A, C]B$,
b) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.
[(a) is similar to the Leibniz rule of derivatives. (b) is called Jacobi identity.]

4- [Byron-Fuller 3.13] Prove that $\mathrm{tr}(AB) = \mathrm{tr}(BA)$ for arbitrary $n \times n$ matrices, A and B. Use this result to prove that if C and D are similar, then $\mathrm{tr}(C) = \mathrm{tr}(D)$.

5- [Byron-Fuller 3.28] Show that the operator

$$T = I + \frac{xD}{1!} + \frac{(xD)^2}{2!} + \cdots + \frac{(xD)^n}{n!}, \tag{5}$$

where $D = d/dt$ [and x is a constant], acts as a translation operator on the space of polynomials (in the variable t) of degree $\leq n$, that is, $Tf(t) = f(t + x)$, if $f(t)$ is in the space of polynomials of degree $\leq n$.