Mathematical Physics

Problem Set 4

due Sunday 12th of Aban in class

1- [Similar to Byron-Fuller 2.9] Consider the action $I[\psi_r, \psi_i] = \int \mathcal{L} dt dx dy dz$ where

$$\mathcal{L} = \frac{\hbar^2}{2m} \nabla\psi^* \cdot \nabla\psi + V\psi^*\psi - \frac{i\hbar}{2}(\dot{\psi}\psi^* - \psi\dot{\psi}^*) \tag{1}$$

is called Lagrangian density and $\psi = \psi_r + i\psi_i$ ($\hbar$ and m are constants, V is a function of x, y, and z, and star denotes complex conjugation). Show that the Euler-Lagrange equations for this action lead to the Schrodinger equation:

$$\left(-\frac{\hbar^2}{2m}\nabla^2 + V\right)\psi = i\hbar\frac{\partial \psi}{\partial t}. \tag{2}$$

2- [Byron-Fuller 2.11]. The Lagrangian density $\mathcal{L}$, which generates a given set of Euler-Lagrange equations, is not unique. Prove this result by showing that adding a divergence to $\mathcal{L}$ does not alter the Euler-Lagrange equations. That is, let

$$\mathcal{L}' = \mathcal{L} + \sum_k \frac{\partial f_k}{\partial x_k} \tag{3}$$

where

$$\mathcal{L} = \mathcal{L}(x_k, \phi_j, \frac{\partial \phi_j}{\partial x_k});$$
$$f_k = f_k(\phi_j);$$
$j = 1, \ldots, m$ indexes the dependent field variables, and
$k = 1, \ldots, n$ indexes the independent variables.

Now prove that $\frac{\delta \mathcal{L}}{\delta \phi_j} = \frac{\delta \mathcal{L}'}{\delta \phi_j}$ where we define

$$\frac{\delta \mathcal{L}}{\delta \phi_j} = \frac{\partial \mathcal{L}}{\partial \phi_j} - \sum_k \frac{\partial}{\partial x_k}\frac{\partial \mathcal{L}}{\partial(\partial \phi_j/\partial x_k)}. \tag{4}$$

3- Let $I = \int f(t, x, \dot{x})dt$ and $I_n = \int f^n(t, x, \dot{x})dt$. Let $x_0(t)$ be a solution of the Euler-Lagrange equation for I and suppose $f(t, x_0(t), \dot{x}_0(t))$ is t-independent. Show that $x_0(t)$ will also satisfy the Euler-Lagrange for I_n. [This is useful in finding geodesics where $f = \sqrt{\dot{x}_1^2 + \dot{x}_2^2 + \cdots}$ can be replaced by f^2, albeit t must be taken as the length parameter along the curve.]

4- Consider a simple pendulum of length l. Find the equations of motion with the Lagrangian method in two ways:

a) Use the angle θ between the pendulum and the vertical direction as the only generalized coordinate: $q = \theta$. Write down the Lagrangian $K - V$ and obtain the equation of motion.

b) Use θ as well as the radial coordinate r (distance from the hanging point to the pendulum mass) as the two generalized coordinates: $q_1 = r$, $q_2 = \theta$; but also employ the constraint $r = l$. Obtain the equations of motion for q_1 and q_2.

[Hint: Your kinetic term is going to involve $\frac{1}{2}l^2\dot{\theta}^2$ in part (a) but in part (b) you have two terms: $\frac{1}{2}(\dot{r}^2 + r^2\dot{\theta}^2)$.]