Mathematical Physics

# Problem Set 3

due Sunday 5th of Aban in class

**1-** Derive the continuity equation for electric charge: Imagine the space is filled with a distribution of moving charges. Let $\rho(\mathbf{x}, t)$ be the scalar field representing the charge density and let $\mathbf{v}(\mathbf{x}, t)$ be the vector field representing the velocity of the charges at point $\mathbf{x}$ and time $t$. Define the current density $\mathbf{J}(\mathbf{x}, t) = \rho(\mathbf{x}, t)\mathbf{v}(\mathbf{x}, t)$ (its magnitude gives the current per unit cross sectional area and its direction is parallel to the electric current). Now follow the steps in Example 1.4 of Byron-Fuller to show that:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0. \tag{1}$$

This means that electric charge is not created nor annihilated anywhere in space; it just goes from one place to another: it's conserved.

**2-** Before Maxwell, "Maxwell's equations" read:

$$\nabla \cdot \mathbf{E} = \rho/\epsilon_0, \qquad \nabla \cdot \mathbf{B} = 0, \qquad \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}, \qquad \nabla \times \mathbf{B} = \mu_0 \mathbf{J}. \tag{2}$$

Maxwell realized that they are inconsistent with the continuity equation (see the previous problem). You can do the same by taking the time derivative of the first equation and adding it to the divergence of the last equation. What modification is required in "Maxwell's equations" above so that the continuity equation holds?

**3-** [Byron-Fuller 2.1] The shortest line (polar coordinates). Find the shortest distance between two points using polar coordinates, i.e., using them as a line element $ds^2 = dr^2 + r^2 d\theta^2$.

**4-** [Byron-Fuller 2.3] Geodesics on a sphere. On a sphere of fixed radius $a$, the line element is given by

$$ds^2 = a^2 d\theta^2 + a^2 \sin^2\theta d\phi^2. \tag{3}$$

Determine the equation of the curve that extremizes the distance between two points. Is the extremum a minimum, a maximum, or what ? [Hint: Let $\theta$ be the independent variable so the integrand of the integral to be extremized is independent of the dependent variable $\phi$.] Now prove: $\phi = \alpha - \sin^{-1}(k \cot\theta)$, where $\alpha$ and $k$ are constants. Then change from the spherical coordinates to Cartesian coordinates to see that this is the equation of a plane through the origin; its intersection with the sphere $r = a$ is the geodesic—a great circle—on the surface of the sphere.

**5-** [Byron-Fuller 2.4] Inverse isoperimetric problem ("isoareametric" problem). Prove that of all simple closed curves enclosing a given area, the least perimeter is possessed by the circle.