Mathematical Physics

Problem Set 2

due Sunday 21st of Mehr in class

1- Prove that $(\mathbf{A}\times\mathbf{B})\cdot(\mathbf{U}\times\mathbf{V}) = (\mathbf{A}\cdot\mathbf{U})(\mathbf{B}\cdot\mathbf{V}) - (\mathbf{A}\cdot\mathbf{V})(\mathbf{B}\cdot\mathbf{U})$.

2- a) Let $\mathbf{F} = f(r)\hat{\mathbf{e}}_\theta$ be a vector field written in cylindrical coordinates. Draw $\mathbf{F}$ on the $z=0$ plane (you can choose your favorite $f(r)$) and find the curl of $\mathbf{F}$ (but this one must be for generic $f(r)$). When does the curl vanish? Does it match your intuition about curl?

b) Repeat part (a) for $\mathbf{F} = f(r)\hat{\mathbf{e}}_r$.

3- [Byron-Fuller 1.6] Prove the vector identities:

a) $\nabla(\mathbf{A}\cdot\mathbf{B}) = (\mathbf{A}\cdot\nabla)\mathbf{B} + (\mathbf{B}\cdot\nabla)\mathbf{A} + \mathbf{A}\times(\nabla\times\mathbf{B}) + \mathbf{B}\times(\nabla\times\mathbf{A})$;

[$(\mathbf{A}\cdot\nabla)\mathbf{B}$ means $A_i\partial_i B_j\hat{\mathbf{e}}_j$ in Cartesian coordinates.]

b) $\nabla\cdot(\mathbf{U}\times\mathbf{V}) = \mathbf{V}\cdot(\nabla\times\mathbf{U}) - \mathbf{U}\cdot(\nabla\times\mathbf{V})$.

4- [Byron-Fuller 1.7] Prove Green's theorem, that is,

$$\int_V (\varphi\nabla^2\phi - \phi\nabla^2\varphi)dV = \int_S (\varphi\nabla\phi - \phi\nabla\varphi)\cdot\mathbf{n}dA. \tag{1}$$

[Hint: Apply Gauss's divergence theorem to the vectors $\mathbf{A} = \varphi\nabla\phi$ and $\mathbf{B} = \phi\nabla\varphi$.]

5- Prove

$$\int_S \mathbf{n}\times\nabla\psi dA = \int_C \psi d\mathbf{s}. \tag{2}$$

[Hint: Apply Stokes's theorem $\int_S(\nabla\times\mathbf{F})\cdot\mathbf{n}dA = \int_C \mathbf{F}\cdot d\mathbf{s}$ to $\mathbf{F} = \psi\hat{\mathbf{e}}_x$, etc.]

6- [optional; not to be graded] Start with

$$\epsilon_{i_1i_2i_3}\det M = \epsilon_{j_1j_2j_3}M_{j_1i_1}M_{j_2i_2}M_{j_3i_3}. \tag{3}$$

(This is really the same as the result of problem 5 in PS1; think about $\det M = \det M^T$.) Multiply both sides by $\left(M^{-1}\right)_{i_3k}$ and sum over i_3.[1] Now show that if M is an orthogonal matrix (i.e., $M^TM = I$) then we have

$$\epsilon_{i_1i_2i_3}M_{ki_3} = \pm\epsilon_{j_1j_2k}M_{j_1i_1}M_{j_2i_2}, \tag{4}$$

depending on whether or not $M\in SO(3)$. Compare with Eq. (1.37) of Byron-Fuller.

[1] This means the i_3k entry of the matrix M^{-1} and we often write it as $M^{-1}_{i_3k}$ without the parentheses. Note that it's different from $\left(M_{i_3k}\right)^{-1} = 1/M_{i_3k}$ which is the inverse of a number, rather than a matrix.