Electrodynamics 1

Problem Set 8

due 1pm on Monday 16th of Day in my office

1- [Jackson 12.1] a) Show that the Lorentz invariant Lagrangian (in the sense of Section 12.1B)

$$-\frac{mU_\alpha U^\alpha}{2} - \frac{q}{c}U_\alpha A^\alpha \tag{1}$$

gives the correct relativistic equations of motion for a particle of mass m and charge q interacting with an external field described by the 4-vector potential $A^\alpha(x)$. [Should the action be $\int Ldt$ or $\int Ld\tau$? ...]

2- Obtain Maxwell's equations from the action

$$S = \int d^4x \left[-\frac{1}{16\pi}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}J_\mu A^\mu \right], \tag{2}$$

just as we did in the class. What will you get if you vary the first term of S with respect to $\mathbf{E}$ and $\mathbf{B}$ instead of A_μ?

3- [Jackson 12.14] An alternative Lagrangian density for the electromagnetic field is

$$\mathcal{L} = -\frac{1}{8\pi}\partial_\alpha A_\beta \partial^\alpha A^\beta - \frac{1}{c}J_\alpha A^\alpha. \tag{3}$$

a) Derive the Euler-Lagrange equations of motion. Are they the Maxwell equations? Under what assumptions?

b) Show explicitly, and with what assumptions, that this Lagrangian density differs from (12.85) by a 4-divergence. Does this added 4-divergence affect the action or the equations of motion?

4- Show that the trace of the electromagnetic energy-momentum tensor is zero, i.e., $\Theta^\mu{}_\mu = 0$.

5- [optional, not to be graded] a) Obtain the Euler-Lagrange equations for the action

$$S = \int d^4x F_{\mu\nu}\mathcal{F}^{\mu\nu}. \tag{4}$$

b) Show that all functions $A_\mu(x)$ satisfy this equation of motion.

c) Prove that $\mathbf{E} \cdot \mathbf{B} = \partial_\mu f^\mu$ where

$$f^\mu = -\frac{1}{2}\left(\mathbf{A} \cdot \mathbf{B}, \phi\mathbf{B} - \mathbf{A} \times \mathbf{E}\right). \tag{5}$$

How can this justify the result of part (b)?