Electrodynamics 1

Problem Set 6

due Saturday 16th of Azar in class

1- [Jackson 6.2] The charge and current densities for a single point charge q can be written formally as

$$\rho(\mathbf{x}',t') = q\delta[\mathbf{x}' - \mathbf{r}(t')]; \qquad \mathbf{J}(\mathbf{x}',t') = q\mathbf{v}(t')\delta[\mathbf{x}' - \mathbf{r}(t')] \tag{1}$$

where $\mathbf{r}(t')$ is the charge's position at time t' and $\mathbf{v}(t')$ is its velocity. In evaluating expressions involving the retarded time, one must put $t' = t_{\mathrm{ret}} = t - R(t')/c$, where $\mathbf{R} = \mathbf{x} - \mathbf{r}(t')$ (but $\mathbf{R} = \mathbf{x} - \mathbf{x}'(t')$ inside the delta functions).

a) As a preliminary to deriving the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge, show that

$$\int d^3x' \delta[\mathbf{x}' - \mathbf{r}(t_{\mathrm{ret}})] = \frac{1}{\kappa} \tag{2}$$

where $\kappa = 1 - \mathbf{v}\cdot\hat{\mathbf{R}}/c$. Note that κ is evaluated at the retarded time.

b) Starting with the Jefimenko generalizations of the Coulomb and Biot-Savart laws, use the expressions for the charge and current densities for a point charge and the result of part (a) to obtain the Heaviside-Feynman expressions for the electric and magnetic fields of a point charge,

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{\kappa R^2}\right]_{\mathrm{ret}} + \frac{\partial}{c\partial t}\left[\frac{\hat{\mathbf{R}}}{\kappa R}\right]_{\mathrm{ret}} - \frac{\partial}{c^2\partial t}\left[\frac{\mathbf{v}}{\kappa R}\right]_{\mathrm{ret}}\right\} \tag{3}$$

and

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R^2}\right]_{\mathrm{ret}} + \frac{\partial}{c\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa R}\right]_{\mathrm{ret}}\right\} \tag{4}$$

c) In our notation Feynman's expression for the electric field is

$$\mathbf{E} = \frac{q}{4\pi\epsilon_0}\left\{\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\mathrm{ret}} + \frac{[R]_{\mathrm{ret}}}{c}\frac{\partial}{\partial t}\left[\frac{\hat{\mathbf{R}}}{R^2}\right]_{\mathrm{ret}} + \frac{\partial^2}{c^2\partial t^2}\left[\hat{\mathbf{R}}\right]_{\mathrm{ret}}\right\} \tag{5}$$

while Heaviside's expression for the magnetic field is

$$\mathbf{B} = \frac{\mu_0 q}{4\pi}\left\{\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa^2 R^2}\right]_{\mathrm{ret}} + \frac{1}{c[R]_{\mathrm{ret}}}\frac{\partial}{\partial t}\left[\frac{\mathbf{v}\times\hat{\mathbf{R}}}{\kappa}\right]_{\mathrm{ret}}\right\} \tag{6}$$

Show the equivalence of the two sets of expressions for the fields.

References: O. Heaviside, *Electromagnetic Theory*, Vol. 3 (1912), p. 464, Eq. (214). R. P. Feynman, *The Feynman Lectures in Physics*, Vol. 1 (1963), Chapter 28, Eq. (28.3).

2- Consider Maxwell's equations with magnetic monopoles. Show that the continuity equation for electric charge (charge conservation) still holds. Show that the magnetic charge also satisfies the continuity equation: $\partial\rho_m/\partial t + \nabla\cdot\mathbf{J}_m = 0$.

3- Consider Maxwell's equations and Lorentz force law with magnetic monopoles. Show that magnetic fields have vanishing work on *electric* charges. Similarly show that electric fields have vanishing work on *magnetic* charges. Follow the derivation of Poynting's equation and explain the difference from the standard (no magnetic monopoles) case.

4- Recall the proof that led to the existence of scalar and vector potentials (section 6.2 in Jackson). Is it still valid in the presence of magnetic monopoles?