Electrodynamics 1

Problem Set 3

due Saturday 11th of Aban in class

1- Prove each equation in the left column and compare it with the corresponding one in the right column where Dirac's bra-ket notation is used:

$$
\begin{aligned}
&-i\frac{\partial}{\partial\phi}Y_l^m(\theta,\phi) = mY_l^m(\theta,\phi) && \text{vs} \quad L_z|l,m\rangle = m\hbar|l,m\rangle \\
&-\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right)+\frac{1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi}\right]Y_l^m(\theta,\phi) = l(l+1)Y_l^m(\theta,\phi) && \text{vs} \quad L^2|l,m\rangle = l(l+1)\hbar^2|l,m\rangle \\
&-ie^{i\phi}\left[i\frac{\partial}{\partial\theta}-\cot\theta\frac{\partial}{\partial\phi}\right]Y_l^m(\theta,\phi) = \sqrt{(l-m)(l+m+1)}Y_l^{m+1}(\theta,\phi) && \text{vs} \quad L_+|l,m\rangle = \sqrt{(l-m)(l+m+1)}\hbar|l,m+1\rangle.
\end{aligned}
\tag{1}
$$

[To gain further intuition you may note that a function on the unit sphere S^2 can be thought of as a vector $|f\rangle$. Then it can be expanded in the $|\theta,\phi\rangle$ basis as

$$|f\rangle = \int d\Omega f(\theta,\phi)|\theta,\phi\rangle, \tag{2}$$

where $f(\theta,\phi) = \langle\theta,\phi|f\rangle$, or equivalently, in the $|l,m\rangle$ basis as

$$|f\rangle = \sum_{l,m} f_{l,m}|l,m\rangle, \tag{3}$$

where $f_{lm} = \langle l,m|f\rangle$. The relation between the two bases comes from $\langle\theta,\phi|l,m\rangle = Y_l^m(\theta,\phi)$, which yields the familiar expansion:

$$f(\theta,\phi) = \sum_{l,m} f_{l,m}Y_l^m(\theta,\phi), \qquad f_{l,m} = \int Y_l^{m*}(\theta,\phi)f(\theta,\phi)d\Omega.] \tag{4}$$

2- Find the Green function for the Dirichlet problem inside a rectangular box defined by the six planes, $x=0$, $y=0$, $z=0$, $x=a$, $y=b$, and $z=c$. Use the method of Section 3.12 in Jackson (i.e., expansion in terms of eigenfunctions of the Laplacian operator).

3- [optional, not to be graded; extra credit] Find the Green function for the Dirichlet problem inside a sphere. Use the method of Section 3.12 in Jackson (i.e., expansion in terms of eigenfunctions of the Laplacian operator). Compare with the known result from the method of images.