Electrodynamics 1

Problem Set 1

due Saturday 6th of Mehr in class

1- [Jackson 1.5] The time-averaged potential of a neutral hydrogen atom is given by

$$\Phi = \frac{q}{4\pi\epsilon_0}\frac{e^{-\alpha r}}{r}\left(1+\frac{\alpha r}{2}\right) \tag{1}$$

where q is the magnitude of the electronic charge, and $\alpha^{-1} = a_0/2$, a_0 being the Bohr radius. Find the distribution of charge (both continuous and discrete) that will give this potential and interpret your result physically.

2- [Jackson 1.10] Prove the mean value theorem: For charge-free space the value of the electrostatic potential at any point is equal to the average of the potential over the surface of any sphere centered on that point.

3- Let $f(z) = u(x,y) + iv(x,y)$ be an analytic complex function of $z = x + iy$.

a) Show that both u and v satisfy the two-dimensional Laplace equation, i.e.,

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0. \tag{2}$$

[Hint: Use the Cauchy-Riemann equations.]

b) Show that the equipotential levels of u are perpendicular to those of v.

c) Find u and v for the function $f(z) = \log z$,[1] draw curves of $u = \text{const.}$ and $v = \text{const.}$ and describe the charge configuration that gives rise to the potential $\Phi = u$.

4- a) Prove the Helmholtz decomposition theorem: Let $\mathbf{F}(\mathbf{x})$ be a vector field on $\mathbb{R}^3$ and V be a volume bounded by S. Then

$$\mathbf{F}(\mathbf{x}) = -\nabla\Phi(\mathbf{x}) + \nabla\times\mathbf{A}(\mathbf{x}), \tag{3}$$

where

$$\begin{aligned}\Phi(\mathbf{x}) &= \frac{1}{4\pi}\int_V \frac{\nabla'\cdot\mathbf{F}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' - \frac{1}{4\pi}\int_S \frac{\mathbf{F}(\mathbf{x}')\cdot\mathbf{n}'da'}{|\mathbf{x}-\mathbf{x}'|},\\ \mathbf{A}(\mathbf{x}) &= \frac{1}{4\pi}\int_V \frac{\nabla'\times\mathbf{F}(\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}d^3x' + \frac{1}{4\pi}\int_S \frac{\mathbf{F}(\mathbf{x}')\times\mathbf{n}'da'}{|\mathbf{x}-\mathbf{x}'|}.\end{aligned} \tag{4}$$

So if for any reason the surface terms vanish, a knowledge of the divergence and the curl of $\mathbf{F}$ at all points in a region is enough to specify $\mathbf{F}$ everywhere in that region.

b) Apply this theorem to electrostatics and obtain Jackson's Eq. (1.36).

5- [optional, not to be graded; Jackson 1.14] Consider the electrostatic Green functions of Section 1.10 for Dirichlet and Neumann boundary conditions on the surface S bounding the volume V. Apply Green's theorem (1.35) with integration variable $\mathbf{y}$ and $\phi = G(\mathbf{x},\mathbf{y})$, $\psi = G(\mathbf{x}',\mathbf{y})$, with $\nabla^2_{\mathbf{y}}G(\mathbf{z},\mathbf{y}) = -4\pi\delta(\mathbf{y}-\mathbf{z})$. Find an expression for the difference $[G(\mathbf{x},\mathbf{x}') - G(\mathbf{x}',\mathbf{x})]$ in terms of an integral over the boundary surface S.

a) For Dirichlet boundary conditions on the potential and the associated boundary condition on the Green function, show that $G_D(\mathbf{x},\mathbf{x}')$ must be symmetric in $\mathbf{x}$ and $\mathbf{x}'$.

b) For Neumann boundary conditions, use the boundary condition (1.45) for $G_N(\mathbf{x},\mathbf{x}')$ to show that $G_N(\mathbf{x},\mathbf{x}')$ is not symmetric in general, but that $G_N(\mathbf{x},\mathbf{x}') - F(\mathbf{x})$ is symmetric in $\mathbf{x}$ and $\mathbf{x}'$, where

$$F(\mathbf{x}) = \frac{1}{S}\oint_S G_N(\mathbf{x},\mathbf{y})da_y. \tag{5}$$

c) Show that the addition of $F(\mathbf{x})$ to the Green function does not affect the potential $\Phi(\mathbf{x})$. See problem 3.26 for an example of the Neumann Green function.

[1] $\log z$ is not analytic everywhere on the complex plane. But you can remove the branch cut $\mathrm{Re}z \leq 0$ so that it becomes analytic.